

Generalized Cayley Graphs and Cellular Automata over them

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Abstract. Cayley graphs have a number of useful features: the ability to graphically represent finitely generated group elements and their equality; to name all vertices relative to a point; the fact that they have a well-defined notion of translation, and that they can be endowed with a compact metric. We propose a notion of graph associated to a language, which conserves or generalizes these features. Whereas Cayley graphs are regular; associated graphs are arbitrary, although of a bounded degree. Moreover, it is well-known that cellular automata can be characterized as the set of translation-invariant continuous functions for a distance on the set of configurations that makes it a compact metric space; this point of view makes it easy to extend their definition from grids to Cayley graphs. Similarly, we extend their definition to these arbitrary, bounded degree, time-varying graphs.

Introduction

Cayley graphs. Cayley graphs are graphs associated to a finite set of generators of a group, together with their inverses. For instance let this set be $\pi = \{a, a^{-1}, b, b^{-1}, \dots\}$. Then the vertices of the graph can be designated by words on π , e.g. $a, a^2, a^{-1}, a.b, \dots$, but more precisely they are the equivalence classes of these words with respect to the group equivalence \equiv , e.g. $b^{-1}.b.a$ and a designate the same vertex. The edges are those pairs $(u, u.a)$. Cayley graphs have been used intensively because they have a number of useful features:

- Once an origin has been chosen, all other vertices can be named relatively to that point.
- The resulting graph represents the group, i.e. the set of terms and their equality.
- There is a well-defined notion of translation of the graph, which corresponds to changing the point representing the origin, or equivalently applying an element of the group to all vertices.

- The set of configurations over a given Cayley graph can be given the structure of a compact metric space, which has been used in order to define Cellular Automata over them.

In this paper, we propose a notion of graph associated to an adjacency language L and its equivalence relation \equiv_L , which conserves or generalizes these features. Whereas Cayley graphs are very regular, associated graphs are arbitrary, albeit connected and of a bounded degree.

Cellular Automata. Cellular Automata (CA) consist of a grid of identical square cells, each of which may take one of a finite number of possible states. The entire array evolves in discrete time steps. The time evolution is required to be translation-invariant (it commutes with translations of the grid) and causal (information cannot be transmitted faster than a fixed number of cells per time step). Whilst Cellular Automata are usually defined as exactly the functions having those physics-like symmetries, it turns out that they can also be characterized in purely mathematical terms as the set of translation-invariant continuous functions [10] for a certain compact metric. As a consequence CA definitions are quite naturally extended from grids to Cayley graphs, where most of the theory carries through [17,3]. Moving on, there have been several approaches to generalize Cellular Automata not just to Cayley graphs, but to arbitrary connected graphs of bounded degree:

- With a fixed topology [16,4,8], in order to describe certain distributed algorithms.
- Through the simulation environments of [7,23,14] which offer the possibility of applying a local rewriting rule simultaneously in different non-conflicting places.
- Through concrete instances advocating the concept of CA extended to time-varying graphs as in [22,13,12], some of which are advanced algorithmic constructions [21,20].
- Through Amalgamated Graph Transformations [2,15] and Parallel Graph Transformations [5,18,19], which work out rigorous ways to apply a local rewriting rule synchronously throughout a graph.

The approach of this paper is different in the sense that it first generalizes Cayley graphs, and then applies the mathematical characterization of Cellular Automata as the set of translation-invariant continuous functions in order to generalize CA. Compared with the above mentioned CA papers, the contribution is to extend the fundamental structure theorems about Cellular Automata to arbitrary, connected, bounded degree, time-varying graphs. Compared with the above mentioned Graph Rewriting papers, the contribution is to deduce aspects of Amalgamated/Parallel Graph Transformations from the axiomatic and topological properties of the global function.

Causal Graph Dynamics. The work by [1] by Dowek and one of the authors already achieves an extension of Cellular Automata to arbitrary, bounded degree, time-varying graphs, also through a notion of continuity, with the same motivations. However, graphs in [1] lack a compact metric over graphs, which is left as

an open question. As a consequence all the necessary facts about the topology of Cayley graphs get reproven. It also leaves open whether the notion of invertible causal graph dynamics is the most general one, and whether causal graph dynamics are computable. *Most of these issues vanish in the new formalism; which suggests that the new formalism itself is the main contribution of this paper.*

This paper. Section 1 provides a generalization of Cayley graphs. This takes the form of an isomorphism between graphs and languages endowed with an equivalence. Section 2 provides basic operations upon generalized Cayley graphs. Section 3 provides facts about the topology of generalized Cayley graphs. It follows that continuous functions are uniformly continuous, composable. Section 4 establishes a notion of Cellular Automata over generalized Cayley graphs. A theorem of equivalence between a mathematical and a constructive approach is given. It also shows that recognizing good local constructions is a recursive task, as well as computing their effect over finite graphs; this grants our model the status of a model of computation. Section 5 provides some examples.

1 Generalized Cayley graphs: definitions

Notations. All graphs are assumed to be connected. The *vertices* of the graphs we consider in this paper are uniquely identified by a name u in V . Vertices may also be labelled with a *state* $\sigma(u)$ in Σ a finite set. Each vertex has *ports* i in π a finite set. A vertex and its port are written $u:a$. An *edge* is an unordered pair $\{u:a, v:b\}$. The port of a node can only appear in one edge, so that the degree of the graphs is always bounded by $|\pi|$. Edges may also be labelled with a *state* $\delta(\{u:a, v:b\})$ in Δ a finite set. All languages defined are on the finite alphabet $\Pi = \pi^2$ with a suffix in $S = \{\varepsilon, 1, 2, 3, \dots\}$, i.e. they are subsets of Π^*S , where ‘.’ represents the concatenation of words and ε the empty word, as usual. The set of vertices V is the set of languages.

1.1 Graphs as paths

Definitions 1 to 4 are as in [1].

Definition 1 (Graph). A graph G is given by

- An at most countable subset $V(G)$ of V , whose elements are called vertices.
- A finite set π , whose elements are called ports.
- A set $E(G)$ of non-intersecting two element subsets of $V(G):\pi$, whose elements are called edges.

The graph is assumed to be connected.

Definition 2 (Labelled graph). A labelled graph is a triple (G, σ, δ) , also denoted simply G when it is unambiguous, where G is a graph, and σ and δ respectively label the vertices and the edges of G :

- σ is a partial function from $V(G)$ to Σ ;

- δ is a partial function from $E(G)$ to Δ .

The set of all graphs with ports π is written \mathcal{G}_π . The set of labelled graphs with states Σ, Δ and ports π is written $\mathcal{G}_{\Sigma, \Delta, \pi}$. To ease notations, we sometimes write $v \in G$ for $v \in V(G)$.

Definition 3 (Pointed graph). A pointed (labelled) graph is a pair (G, p) with $p \in G$. The set of pointed graphs with ports π is written \mathcal{P}_π . The set of pointed labelled graphs with states Σ, Δ and ports π is written $\mathcal{P}_{\Sigma, \Delta, \pi}$.

Definition 4 (Isomorphism). An isomorphism R is a function from \mathcal{G}_π to \mathcal{G}_π which is specified by a bijection $R(\cdot)$ from V to V . The image of a graph G under the isomorphism R is a graph RG whose set of vertices is $R(V(G))$, and whose set of edges is $\{\{R(u) : a, R(v) : b\} \mid \{u : a, v : b\} \in E(G)\}$. Similarly, the image of a pointed graph $P = (G, p)$ is the pointed graph $RP = (RG, R(p))$. When P and Q are isomorphic we write $P \approx Q$, defining an equivalence relation on the set of Graphs. The definition extends to pointed labelled graphs.

In the particular graphs we are considering, the vertices can be uniquely distinguished by the paths that lead to them starting from the pointer vertex. Hence, we might just as well forget about vertex names.

Definition 5 (Pointed graph modulo). Let P be a pointed (labelled) graph (G, p) . The pointed (labelled) graph modulo \tilde{P} is the equivalence class of P with respect to the equivalence relation \approx . The set of pointed graphs modulo with ports π is written $\tilde{\mathcal{P}}_\pi$. The set of pointed labelled graphs modulo with states Σ, Δ and ports π is written $\tilde{\mathcal{P}}_{\Sigma, \Delta, \pi}$.

Given such a pointed graph modulo, its set of paths forms a language, endowed with a notion of equivalence whenever two paths designate the same vertex. The language, together with its equivalence, is referred to as a the path structure.

Definition 6 (Language of paths). Given a pointed graph modulo \tilde{P} , we say that u is a path of \tilde{P} : there is a sequence u of ports $a_i b_i$ such that, starting from the pointer, it is possible to travel in the graph according to this sequence. We define the language of paths $L(\tilde{P})$ of \tilde{P} as the set of these paths. More formally, $u \in L(\tilde{P})$ if and only if there exists $(G, p) \in \tilde{P}$ and $v_1, \dots, v_{|u|} \in V(G)$ such that for all $i \in \{0 \dots |u| - 1\}$, one has $\{v_i : a_i, v_{i+1} : b_i\} \in E(G)$, with $v_0 = p$ and $u_i = a_i b_i$.

Definition 7 (Equivalence of paths). Given a pointed graph modulo \tilde{P} , we define the equivalence of paths relation $\equiv_{\tilde{P}}$ on $L(\tilde{P})$ such that for all paths $u, u' \in L(\tilde{P})$, $u \equiv_{\tilde{P}} u'$ if and only if, starting from the pointer, u and u' lead to the same vertex of \tilde{P} . More formally, $u \equiv_{\tilde{P}} u'$ if and only if there exists $(G, p) \in \tilde{P}$ and $v_1, \dots, v_{|u|}, v'_1, \dots, v'_{|u'|} \in V(G)$ such that for all $i \in \{0 \dots |u| - 1\}$, $i' \in \{0 \dots |u'| - 1\}$, one has $\{v_i : a_i, v_{i+1} : b_i\} \in E(G)$, $\{v'_{i'} : a'_{i'}, v'_{i'+1} : b'_{i'}\} \in E(G)$, with $v_0 = p$, $v'_0 = p$, $u_i = a_i b_i$, $u'_{i'} = a'_{i'} b'_{i'}$ and $v_{|u|} = v'_{|u'|}$.

Definition 8 (Path structure). Given a pointed graph modulo \tilde{P} , we define the structure of paths $X(\tilde{P})$ as the structure $\langle L(\tilde{P}), \equiv_{\tilde{P}} \rangle$. The set of all path structures is the set $\{X(\tilde{P}) \mid \tilde{P} \in \mathcal{P}_\pi\}$. It is written $X(\tilde{\mathcal{P}}_\pi)$.

Given two pointed graphs modulo, any difference between them shows up in their path structure.

Proposition 1 (Pointed graphs modulo and path structures isomorphism). The function $\tilde{P} \mapsto X(\tilde{P})$ is a bijection between $\tilde{\mathcal{P}}_\pi$ and $X(\tilde{\mathcal{P}}_\pi)$.

Proof. [Surjectivity]. By definition of $X(\tilde{\mathcal{P}}_\pi)$.

[Injectivity]. Let us suppose that $X(\tilde{P}) = X(\tilde{Q})$. Then $\equiv_{\tilde{P}}$ and $\equiv_{\tilde{Q}}$ must have the same number of equivalence classes and $|V(\tilde{P})| = |V(\tilde{Q})|$. Let us choose two graphs $P \in \tilde{P}$ and $Q \in \tilde{Q}$. For any vertex u of P , there is a unique equivalence class c of $\equiv_{\tilde{P}}$ such that the paths of c lead to u in P . Since $\equiv_{\tilde{P}}$ and $\equiv_{\tilde{Q}}$ are supposed equal, c is also an equivalence class of $\equiv_{\tilde{Q}}$. Conversely given c an equivalence class of $\equiv_{\tilde{Q}}$, there is a unique v of Q such that the paths of c lead to v in Q . Then, the paths which point to u in P are the same as those which point to v in Q . We can now safely define a function R over the set of names such that each vertex u in P has image its corresponding vertex v in Q , and extend it to the entire set V with the identity. Let us consider two vertices u and u' in P linked by an edge $\{u : i, u' : j\}$ and their corresponding vertices v and v' in Q . As $P \in \tilde{P}$, we have that the equivalence classes $\tilde{u}.ij = \tilde{u}'$. As the classes representing v and v' are equal to $\tilde{u}.ij\tilde{u}'$. Thus R is a graph isomorphism, and P and Q are isomorphic. This is true for every $P \in \tilde{P}$ and $Q \in \tilde{Q}$ thus $\tilde{P} = \tilde{Q}$.

1.2 Paths as languages

Inversely, we could have started by defining a certain class of languages endowed with an equivalence, namely adjacency structures, and then asked whether the path structures of graph modulo fall into this class. This is the purpose of the following definitions and lemma.

Definition 9 (Completeness). Let $L \subseteq \Pi^*$ be a language and \equiv_L an equivalence on this language. The tuple (L, \equiv_L) is said to be complete if and only if

- $\forall u, v \in \Pi^* \quad u.v \in L \Rightarrow u \in L$
- $\forall u, u' \in L \forall v \in \Pi^* \quad (u \equiv_L u' \wedge u.v \in L) \Rightarrow (u'.v \in L \wedge u'.v \equiv_L u.v)$
- $\forall u \in L \forall a, b \in \pi \quad u.ab \in L \Rightarrow (u.ab.ba \in L \wedge u.ab.ba \equiv_L u)$

Definition 10 (Adjacency structure). Let $L \subseteq \Pi^*$ be a language and \equiv_L an equivalence on this language. The tuple (L, \equiv_L) defines an adjacency structure if and only if it is complete and

$$\forall u, u' \in L \forall a, b, c \in \pi \quad (u \equiv_L u' \wedge u.ab \in L \wedge u'.ac \in L) \Rightarrow b = c.$$

When this is the case, L is referred to as an adjacency language and \equiv_L as an adjacency equivalence. The adjacency structure X is denoted $\langle L, \equiv \rangle$. The set of all adjacency structures X is written \mathcal{X}_π .

Definition 11 (Associated (pointed) graph (modulo)). Let X be some adjacency structure $\langle L, \equiv_L \rangle$. Let $V(X)$ be the set of equivalence classes of X . Let $P(X)$ be the pointed graph $(G(X), \tilde{\varepsilon})$, with $G(X)$ such that:

- The set of vertices $V(G(X))$ is $V(X)$;
- The edge $\{\tilde{u}:a, \tilde{v}:b\}$ is in $E(G(X))$ if and only if $u.ab \equiv_L v$, for all $u \in \tilde{u}$ and $v \in \tilde{v}$.

We define the associated graph to be $G(X)$. We define the associated pointed graph to be $P(X)$. We define the associated pointed graph modulo to be $\tilde{P}(X)$.

Soundness: The properties of adjacency structures ensure that the ports of the vertices are not used several times and that the graph is connected.

Definition 12 (Labelled adjacency structure). Let $X = \langle L, \equiv_L \rangle$ be an adjacency structure. A labelling with states Σ, Δ is given by a labelling for $\tilde{P}(X)$. The set of labelled adjacency structures with states Σ, Δ and ports π is written $\mathcal{X}_{\Sigma, \Delta, \pi}$.

Lemma 1 (Path structures are adjacency structures). Let \tilde{P} be a pointed graph modulo. Then $X(\tilde{P})$ is an adjacency structure. Hence $X(\tilde{\mathcal{P}}_\pi) \subseteq \mathcal{X}_\pi$.

Proof. [Completeness]. If $u.v$ is a valid path in \tilde{P} , then the truncated path u is a valid path in \tilde{P} and belongs to $L(\tilde{P})$.

If two paths u and v in \tilde{P} lead to the same vertex, i.e. $u \equiv_{\tilde{P}} v$, then extending u and v by the same path w will still lead to the same vertex i.e. if $u.w \in L(\tilde{P})$ $u.w \equiv_{\tilde{P}} v.w$.

If $u.ab$ is a valid path in \tilde{P} then the extension $u.ab.ba$ consisting in going back on the last visited vertex is still a valid path and leads to the vertex pointed by u .

Summarizing, the completeness properties are verified by construction of the language of path $L(\tilde{P})$ and the relation $\equiv_{\tilde{P}}$.

[Adjacency structure]. Let us consider two paths u and v in $L(\tilde{P})$ and three ports a, b, c such that $u \equiv_{\tilde{P}} v$ and $u.ab \equiv_{\tilde{P}} u.ac$. Then, for the graph \tilde{P} to be well defined we have that $b = c$.

Not only do we have that path structures are adjacency structures, but it also turns out that any adjacency structure can be generated this way, i.e. it is the path structure of some pointed graph modulo.

Proposition 2 (Adjacency structures are path structures). Let X be some adjacency structure. The equality $X = X(\tilde{P}(X))$ holds. Hence $\mathcal{X}_\pi = X(\tilde{\mathcal{P}}_\pi)$.

Proof. Let $X = \langle L, \equiv_L \rangle$ and $X(\tilde{P}(X)) = \langle L', \equiv_{L'} \rangle$.

$[X \subseteq X(\tilde{P}(X))]:$

Let us consider $w \in L$. By construction of $\tilde{P}(X)$, there exists a path w in $\tilde{P}(X)$. By definition of the function X , we have that this path will be represented

by the word $w \in L'$. Now, let us consider two words u and v in L such that $u \equiv v$. By construction of $\tilde{P}(X)$, u and v will be two paths of $\tilde{P}(X)$ leading to the same vertex. By definition of the function X , the two words u and v in L' will be equivalent regarding to the relation \equiv' .

$[X(\tilde{P}(X)) \subseteq X]$:

Suppose there exists $w \in L'$ but $w \notin L$. Let $u.ab$ be the longest prefix of w such that $u \in L$ but $u.ab \notin L$. $u.ab \in L'$ implies that there exists some edge from port a of u to port b of $u.ab$ in $\tilde{P}(X)$ which is, by construction of $\tilde{P}(X)$, equivalent to $u.ab \in L$. Hence a contradiction.

Now assume that there exist two words $u, v \in L'$ such that $u \equiv' v$ and such that $u \not\equiv v$. As $u \not\equiv v$, u and v are two paths in $\tilde{P}(X)$ leading to two different vertices of $\tilde{P}(X)$. By definition of the function X , these two paths are represented by the words u and v in $X(\tilde{P}(X))$ and are not equivalent, hence contradiction.

1.3 Graphs as languages

Generalized Cayley graphs. The following result comes out as a corollary of Propositions 1 and 2:

Theorem 1 (Pointed graphs modulo and adjacency structures isomorphism). *The function $\tilde{P} \mapsto X(\tilde{P})$ is a bijection between $\tilde{\mathcal{P}}_\pi$ and \mathcal{X}_π , whose inverse is the function $X \mapsto \tilde{P}(X)$. It can be extended into a bijection between $\tilde{\mathcal{P}}_{\Sigma, \Delta, \pi}$ and $\mathcal{X}_{\Sigma, \Delta, \pi}$.*

Conventions. This theorem justifies the fact that

- a (labelled) pointed graph modulo $\tilde{P}(X)$ (resp. \tilde{P}),
- a (labelled) adjacency structure X (resp. $X(\tilde{P})$),
- and their associated graph $G(X)$ (resp. $G(X(\tilde{P}))$)

can be viewed as three presentations of the same mathematical object. Together with Definitions 8 and 11, it also justifies the fact that the vertices of this mathematical object can be designated by

- \tilde{u} an equivalence class of X (resp. $X(\tilde{P})$), i.e. the set of all paths leading to this vertex starting from $\tilde{\varepsilon}$,
- or more directly by u an element of an equivalence class \tilde{u} of X (resp. $X(\tilde{P})$), i.e. a particular path leading to this vertex starting from ε .

These two remarks lead to the following mathematical conventions, which we adopt for convenience. From now on:

- $\tilde{\mathcal{P}}_{\Sigma, \Delta, \pi}$ and $\mathcal{X}_{\Sigma, \Delta, \pi}$ will no longer be distinguished. The latter notation will be preferred. We shall speak of a “generalized Cayley graph” X in $\mathcal{X}_{\Sigma, \Delta, \pi}$.
- \tilde{u} and u will no longer be distinguished. The latter notation will be given the meaning of the former. I.e. we shall speak of a “vertex” u in $V(X)$ (or simply $u \in X$).

- It follows that ‘ \equiv ’ and ‘ $=$ ’ will no longer be distinguished. The latter notation will be given the meaning of the former. I.e. we shall speak of “equality of vertices” $u = v$ (when strictly speaking we just have $\tilde{u} = \tilde{v}$).

Such conventions may seem shocking at first but are very common in algebraic structures, for instance it is common to write $2 + 2 = 4$ even though they are syntactically distinct. In any case, we will make sure that a rigorous meaning can always be recovered by placing tildes back.

Discussion. Clearly this mathematical object, namely adjacency structures or pointed graphs modulo, extends the notion of Cayley graph. In order to recover these, we can first consider adjacency structures $\langle M^*, \equiv_{M^*} \rangle$ where $M = \{aa^{-1}, a^{-1}a \mid a \in \pi\}$. However, the Petersen graph, for instance, can be endowed with such an adjacency structure, while being famously not a Cayley graph. In order to recover exactly the notion of a Cayley graph, we have to impose that \equiv_{M^*} be derived from a group law. Whereas Cayley graphs are highly symmetric, *generalized* Cayley graphs (whose labels are dropped) may be arbitrary connected graphs of bounded degree. This extension is an advantageous one, since all of the key features of Cayley graphs remain: We are able to name vertices relative to a point, through the word describing the path from that point, and in fact the topology of the graph describes the equivalence structure upon words. We have a well-defined notion of translation, which is described as part of the basic operations upon these graphs in Section 2. We can define a distance between these graphs, which makes $\mathcal{X}_{\Sigma, \Delta, \pi}$ a compact metric space, as done in Section 3.

2 Generalized Cayley graphs: basic operations

For a pointed graph (G, p) non-modulo:

- the neighbours of radius r are just those vertices which can be reached in r steps starting from the pointer p ;
- the disk of radius r , written G_p^r , is the subgraph induced by the neighbours of radius $r + 1$, with labellings restricted to the neighbours of radius r and the edges between them, and pointed at p .

Notice that the vertices of G_p^r continue to have the same names as they used to have in G . See [1] for details. For generalized Cayley graphs the analogous operation is:

Definition 13 (Disk). *Let $X \in \mathcal{X}_{\Sigma, \Delta, \pi}$ be a generalized Cayley graph and (G, ε) its associated pointed graph. Let X^r be $X(\tilde{G}_\varepsilon^r)$. The generalized Cayley graph $X^r \in \mathcal{X}_{\Sigma, \Delta, \pi}$ is referred to as the disk of radius r of X . The set of disks of radius r with states Σ , Δ and ports π is written $\mathcal{X}_{\Sigma, \Delta, \pi}^r$.*

Notice that the vertices of X^r no longer have quite the same names as they used to have in X . Indeed, in a generalized Cayley graph, vertices are designated by those paths that lead to them, starting from the vertex ε , and there were many

more such paths in X than there are in X^r . Still, it is clear that there is a natural inclusion $V(X^r) \subseteq V(X)$. Thus, we will commonly say that a vertex of $u \in X^r$ belongs to X , even though technically we are referring to the unique vertex u' of X such $u \subseteq u'$. Similarly, we will commonly say that a vertex of $u' \in X$ belongs to X^r when we actually mean that there is a unique vertex u of X^r such that $u \subseteq u'$.

Definition 14 (Size). Let $X \in \mathcal{X}_{\Sigma, \Delta, \pi}$ be a generalized Cayley graph. We say that a vertex $u \in X$ has size less or equal to $r + 1$, and write $|u| \leq r + 1$, if and only if $u \in X^r$. We denote $V(\mathcal{X}_\pi^r) = \bigcup_{X \in \mathcal{X}_\pi^r} V(X)$.

It will help to have a notation for the graph where vertices are named relatively to some other vertex u .

Definition 15 (Shift). Let $X \in \mathcal{X}_{\Sigma, \Delta, \pi}$ be a generalized Cayley graph and (G, ε) its associated pointed graph. Consider $u \in X$ or X^r for some r , let X_u be $X^{\sim}(G, u)$. The generalized Cayley graph X_u is referred to as X shifted by u .

The composition of a shift, and then a restriction, applied on X , will simply be written X_u^r . Whilst this is the analogous operation to G_u^r over pointed graphs non-modulo, notice that the shift-by- u completely changes the names of the vertices of X_u^r . As the naming has become relative to u , the disk X_u^r holds no information about its prior location, u .

We may also want to designate a vertex v by those paths that lead to the vertex u relative to ε , followed by those paths that lead to v relative to u .

Definition 16 (Concatenation). Let $X \in \mathcal{X}_\pi$ be a generalized Cayley graph and (G, ε) its associated pointed graph. Consider $u \in X$ and $v \in X_u$ or X_u^r for some r . Let (G', ε) be the associated pointed graph of $(X_u)_v$, R be an isomorphism such that $G' = RG$, and $u.v$ be $R^{-1}(\varepsilon)$. The vertex $u.v \in X$ is referred to as u concatenated with v .

According to Definition 15, G' and G are isomorphic. Moreover, the restriction of R^{-1} to $V(G')$ is uniquely determined; hence the definition is sound. Notice also that this definition of concatenation coincides with the one that is induced by the concatenation of words belonging to the classes u and v .

It also helps to have a notation for the paths to ε relative to u .

Definition 17 (Inverse). Let $X \in \mathcal{X}_\pi$ be a generalized Cayley graph and (G, ε) its associated pointed graph. Consider $u \in X$. Let (G', ε) be the associated pointed graph of X_u , R be an isomorphism such that $G' = RG$, and \bar{u} be $R(\varepsilon)$. The vertex $\bar{u} \in X_u$ is referred to as the inverse of u .

Notice the following easy facts: $(X_u)_v = X_{u.v}$, $u.\bar{u} = \varepsilon$. Notice also that the isomorphism R such that $G(X_u) = RG(X)$ maps v to $\bar{u}.v$. This last property suggests that we may define shifts upon graphs (non-modulo) as a certain class of isomorphisms. In order to formalize this notion within the set of graphs without appealing to graphs modulo, we demand that the vertices of our graphs be of a particular form.

Definition 18 (Shift isomorphism). Let $X \in \mathcal{X}_\pi$ be a generalized Cayley graph. Let $G \in \mathcal{G}_\pi$ be a graph that has vertices that are disjoint subsets of $V(X).S$ or $V(X^r).S$ for some r . Consider $u \in X$. Let R the isomorphism from $V(X).S$ to $V(X_u).S$ mapping $v.z \mapsto \bar{u}.v.z$, for any $v \in V(X)$ or $V(X^r)$, $z \in S$. Extend this bijection pointwise to act over subsets of $V(X).S$, and let $\bar{u}.G$ to be RG . The graph $\bar{u}.G$ has vertices that are disjoint subsets of $V(X_u).S$, it is referred to as G shifted by u . The definition extends to labelled graphs.

Notice that $G(X_u) = \bar{u}.G(X)$.

The next two definitions are standard, as they give a well-defined meaning to the notion of union of graphs. See [1]. Although here again the definition is given relative to some X .

Definition 19 (Consistency). Let $X \in \mathcal{X}_\pi$ be a generalized Cayley graph. Let G be a labelled graph (G, σ, δ) , and G' be a labelled graph (G', σ', δ') , each one having vertices that are pairwise disjoint subsets of $V(X).S$. The graphs are said to be consistent if and only if:

- $\forall x \in G \forall x' \in G' \quad x \cap x' \neq \emptyset \Rightarrow x = x'$,
- $\forall x, y \in G \forall x', y' \in G' \forall a, a', b, b' \in \pi \quad (\{x:a, y:b\} \in E(G) \wedge \{x':a', y':b'\} \in E(G') \wedge x = x' \wedge a = a') \Rightarrow (b = b' \wedge y = y')$,
- $\forall x, y \in G \forall x', y' \in G' \forall a, b \in \pi \quad x = x' \Rightarrow \delta(\{x:a, y:b\}) = \delta'(\{x':a, y':b\})$ when both are defined,
- $\forall x \in G \forall x' \in G' \quad x = x' \Rightarrow \sigma(x) = \sigma'(x')$ when both are defined.

They are said to be trivially consistent if and only if there is no $x \in G$, $x' \in G'$ such that $x = x'$.

In practice (especially in Section 5), elements of $V(X)$, which were constructed classes of equivalence of words, will often be designated by one of this words. Beware that equality is then tested not on words, but on their equivalence class in $V(X)$, in particular when we require the vertices of G , in the hypotheses of this definition, to be pairwise disjoint subsets of $V(X).S$.

Definition 20 (Union). Let $X \in \mathcal{X}_\pi$ be a generalized Cayley graph. Let G be a labelled graph (G, σ, δ) , and G' be a labelled graph (G', σ', δ') , each one having vertices that are pairwise disjoint subsets of $V(X).S$. Whenever they are consistent, their union is defined. The resulting graph $G \cup G'$ is the labelled graph with vertices $V(G) \cup V(G')$, edges $E(G) \cup E(G')$, labels that are the union of the labels of G and G' .

3 Generalized Cayley graphs: topological properties

Having a well-defined notion of disks allows us to define a topology upon $\mathcal{X}_{\Sigma, \Delta, \pi}$, which is the natural generalization of the well-studied Cantor metric upon CA configurations [10].

Definition 21 (Gromov-Hausdorff-Cantor metrics). Consider the function

$$\begin{aligned} d : \mathcal{X}_{\Sigma, \Delta, \pi} \times \mathcal{X}_{\Sigma, \Delta, \pi} &\longrightarrow \mathbb{R}^+ \\ (X, X') &\mapsto d(X, Y) = 0 \quad \text{if } X = Y \\ (X, X') &\mapsto d(X, Y) = 1/2^r \quad \text{otherwise} \end{aligned}$$

where r is the minimal radius such that $X^r \neq Y^r$.

The function $d(.,.)$ is such that for $\epsilon > 0$ we have (with $r = \lfloor -\log_2(\epsilon) \rfloor$):

$$d(X, Y) < \epsilon \Leftrightarrow X^r = Y^r.$$

It defines an ultrametric distance.

Soundness: [Nonnegativity, symmetry, identity of indiscernibles] are obvious.
[Equivalence]

$$\begin{aligned} d(X, X') < \epsilon &\Leftrightarrow d(X, Y) = 1/2^k \text{ with } k \in \mathbb{N} \wedge 1/2^k < \epsilon \\ &\Leftrightarrow k = \min\{r \in \mathbb{N} \mid X^r \neq Y^r\} \wedge 1/2^k < \epsilon \\ &\Leftrightarrow_{r=k-1} X^r = Y^r \text{ with } r \in \mathbb{N} \wedge 1/2^{r+1} < \epsilon \\ &\Leftrightarrow X^r = Y^r \text{ with } r = \lfloor -\log_2(\epsilon) \rfloor. \end{aligned}$$

[Ultrametricity] Consider k such that $1/2^k = d(X, Z)$ and l such that $1/2^l = d(X, Y)$. By definition of the metric X, Z differ only after index k and X, Y differ only after index l . Suppose $k \leq l$ so that Y, Z differ only after index k . But then $d(Y, Z) = 1/2^k$ which is $d(X, Z)$.

[Triangle inequality] is obvious from the ultrametricity.

The fact that generalized Cayley graphs are pointed graphs modulo, i.e. the fact that they have no “vertex name degree of freedom” is key to proving the following property. Indeed, compactness crucially relies on the set being “finite-branching”, meaning that the set of possible graphs, as one progressively enlarges the radius of a disk, remains finite. This does not hold for usual graphs.

Lemma 2 (Compactness). $(\mathcal{X}_{\Sigma, \Delta, \pi}, d)$ is a compact metric space, i.e. every sequence admits a converging subsequence.

Proof. This is essentially König’s Lemma. Let us consider an infinite sequence of graphs $(X(n))_{n \in \mathbb{N}}$. Because Σ and Δ are finite, and there is an infinity of elements of $(X(n))$, there must exist a graph of radius zero X^0 such that there is an infinity of elements of $(X(n))$ fulfilling $X(n)^0 = X^0$. Choose one of them to be $X(n_0)$, i.e. $X(n_0)^0 = X^0$. Now iterate: because the degree of the graph is bounded by π , and because Σ and Δ are finite but there is an infinity of elements of $(X(n))$ having the above property, there must exist a pointed graph of radius one X^1 such that $(X^1)^0 = X^0$ and such that there is an infinity of elements of $(X(n))$ having $X(n)^1 = X^1$. Choose one of them as $X(n_1)$, i.e. $X(n_1)^1 = X^1$. Etc. The limit is the unique graph X' having disks $X'^k = X^k$ for all k .

Continuity is a crucial notion in mathematics and CA theory [10].

Definition 22 (Continuous function). *Let (\mathcal{X}, d) be a metric space. A function F from \mathcal{X} to \mathcal{X} is continuous if*

$$\forall X \in \mathcal{X} \forall \epsilon > 0 \exists \eta > 0 \forall Y \in \mathcal{X} \quad d(X, Y) < \eta \Rightarrow d(F(X), F(Y)) < \epsilon.$$

Uniform continuity, on the other hand, is a crucial notion in physics, as it captures the fact that information does not propagate too fast.

Definition 23 (Uniformly continuous function). *Let (\mathcal{X}, d) be a metric space. A function $F : \mathcal{X} \rightarrow \mathcal{X}$ is uniformly continuous if*

$$\forall \epsilon > 0 \exists \eta > 0 \forall X, Y \in \mathcal{X} \quad d(X, Y) < \eta \Rightarrow d(F(X), F(Y)) < \epsilon.$$

Uniform continuity implies continuity, but the converse is not generally true. However, these notions are known to be equivalent in the presence of compactness.

Proposition 3 (Topology recap.).

Let \mathcal{X} and \mathcal{Y} be metric spaces and $F : \mathcal{X} \rightarrow \mathcal{Y}$ continuous. If \mathcal{X} is compact, then F is uniformly continuous.

Theorem 3 is a well-known result in general topology [6]. Their implications for Cellular Automata were first studied in [10], with self-contained elementary proofs available in [11]. For Cellular Automata over Cayley graphs a complete reference is [3]. For Causal Graph Dynamics [1], these had to be reproven by hand, due to the lack of a clear topology in the set of graphs that was considered. Here we are able to rely on the topology of generalized Cayley graphs and reuse Theorem 3 out-of-the-box, which makes the setting of generalized Cayley graphs a very attractive one in order to generalize CA.

4 Causality and Localizability

Causality. The notion of causality we will propose extends the mathematical definition of Cellular Automata over grids and Cayley graphs. The extension is a strict one: not only the graphs have become arbitrary, but they can also vary in time.

The main difficulty we encountered when elaborating an axiomatic definition of causality from $\mathcal{X}_{\Sigma, \Delta, \pi}$ to $\mathcal{X}_{\Sigma, \Delta, \pi}$, was the need to establish a correspondence between the vertices of a generalized Cayley graph X , and those of its image $F(X)$. Indeed, on the one hand it is important to know that a given $u \in X$ has become $u' \in F(X)$, e.g. in order to express shift-invariance $F(X_u) = F(X)_{u'}$. But on the other hand since u' is named relative to ε , its determination requires a global knowledge of X .

The following analogy provides a useful way of tackling this issue. Say that we were able to place a white stone on the vertex $u \in X$ that we wish to follow across evolution F . Later, by observing that the white stone is found at

$u' \in F(X)$, we would be able to conclude that u has become u' . This way of grasping the correspondence between an image vertex and its antecedent vertex is a local, operational notion of an observer moving across the dynamics. But this notion of observer can also be axiomatized, and this is what we have chosen to do.

Definition 24 (Dynamics). *A dynamics (F, R_\bullet) is given by*

- a function $F : \mathcal{X}_{\Sigma, \Delta, \pi} \rightarrow \mathcal{X}_{\Sigma, \Delta, \pi}$;
- a map R_\bullet , with $R_\bullet : X \mapsto R_X$ and $R_X : V(X) \rightarrow V(F(X))$.

Equivalently, we may think of a dynamics (F, R_\bullet) as just one function $F' : \mathcal{X}_{\Sigma', \Delta, \pi} \rightarrow \mathcal{X}_{\Sigma', \Delta, \pi}$ over an extended $\Sigma' = \Sigma \times \{0, 1\}$ indicating whether a vertex is ‘marked with a white stone’ or not. The intuition is that R_X indicates which vertex $R_X(u) \in F(X)$ will end up being marked as a consequence of $u \in X$ being marked.

Definition 25 (Shift-invariance). *A dynamics (F, R_\bullet) is said to be shift-invariant if and only if for every X and $u \in X$, $v \in X_u$,*

- $F(X_u) = F(X)_{R_X(u)}$
- $R_X(u.v) = R_X(u).R_{X_u}(v)$.

The second condition expresses the shift-invariance of R_\bullet . Notice that in the $F' : \mathcal{X}_{\Sigma', \Delta, \pi} \rightarrow \mathcal{X}_{\Sigma', \Delta, \pi}$ point-of-view, the two above conditions would be captured by just one: $F'(X_u) = F'(X)_{R_X(u)}$. Notice also that a shift-invariant dynamics (F, R_\bullet) must fulfill $R_X(\varepsilon) = R_X(\varepsilon).R_X(\varepsilon)$; hence $R_X(\varepsilon) = \varepsilon$.

Definition 26 (Continuity). *A dynamics (F, R_\bullet) is said to be continuous if and only if:*

- $F : \mathcal{X}_{\Sigma, \Delta, \pi} \rightarrow \mathcal{X}_{\Sigma, \Delta, \pi}$ is continuous,
- For all X , for all m , there exists n such that for all X' , $X'^m = X^n$ implies $\text{dom } R_{X'}^m \subseteq X'^n$, $\text{dom } R_X^m \subseteq X^n$ and $R_{X'}^m = R_X^m$.

where R_X^m denotes the partial map obtained as the restriction of R_X to the codomain $F(X)^m$, using the natural inclusion of $F(X)^m$ into $F(X)$.

The second condition expresses the continuity of R_\bullet . Notice that in the $F' : \mathcal{X}_{\Sigma', \Delta, \pi} \rightarrow \mathcal{X}_{\Sigma', \Delta, \pi}$ point-of-view, the two above conditions would be captured by just one: F' continuous. Notice also that since continuity implies uniform continuity upon the compact space $\mathcal{X}_{\Sigma', \Delta, \pi}$, it follows that F' is uniformly continuous, and thus that the second condition can be reinforced into: for all m , there exists n such that for all X, X' , $X'^m = X^n$ implies $R_X^m = R_{X'}^m$. We need one third, last condition:

Definition 27 (Boundedness). *A dynamics (F, R_\bullet) from $\mathcal{X}_{\Sigma, \Delta, \pi}$ to $\mathcal{X}_{\Sigma, \Delta, \pi}$ is said to be bounded if and only if there exists a bound b such that for all X , for all $w' \in F(X)$, there exist $u' \in \text{im } R_X$ and $v' \in F(X)_{u'}^b$ such that $w' = u'.v'$.*

The following is our main definition:

Definition 28 (Causal dynamics). *A dynamics is causal if it is shift-invariant, continuous and bounded.*

Lemma 3 (Bounded inflation). *Consider a causal dynamics F from $\mathcal{X}_{\Sigma,\Delta,\pi}$ to $\mathcal{X}_{\Sigma,\Delta,\pi}$. There exists a bound b such that for all X and $u \in X^r$, we have $|R_X(u)| \leq rb + 1$.*

Proof. Let $ac \in \Pi$, and let E the subset of $\mathcal{X}_{\Sigma,\Delta,\pi}$ of those X such that $ac \in X$. E is closed — any sequence of elements of E converging in $\mathcal{X}_{\Sigma,\Delta,\pi}$ converges in E — and $\mathcal{X}_{\Sigma,\Delta,\pi}$ is compact, therefore E is compact. By continuity modulo, the function $X \mapsto |R_X(ac)|$ is continuous from E to \mathbb{N} ; since E is compact, it must be bounded. The result then follows from the triangle inequality and shift-invariance.

Localizability. The notion of localizability captures the exact same idea as the constructive definition of Cellular Automata, namely a single local rule f applied synchronously and homogeneously across the input graph.

Definition 29 (Dynamics non-modulo). *A function f from $\mathcal{X}_{\Sigma,\Delta,\pi}^r$ to $\mathcal{G}_{\Sigma,\Delta,\pi}$ is said to be a dynamics if and only if for all X the vertices of $f(X)$ are disjoint subsets of $V(X).S$ and $\varepsilon \in f(X)$.*

Definition 30 (Boundedness non-modulo). *A function f from $\mathcal{X}_{\Sigma,\Delta,\pi}^r$ to $\mathcal{G}_{\Sigma,\Delta,\pi}$ is said to be bounded if and only if for all X , the graph $f(X)$ is finite.*

Definition 31 (Local rule). *A function f from $\mathcal{X}_{\Sigma,\Delta,\pi}^r$ to $\mathcal{G}_{\Sigma,\Delta,\pi}$ is a local rule if and only if it is a bounded dynamics and*

- *For any disk X^{r+1} and any $u \in X^0$ we have that $f(X^r)$ and $u.f(X_u^r)$ are non-trivially consistent.*
- *For any disk X^{3r+2} and any $u \in X^{2r+1}$ we have that $f(X^r)$ and $u.f(X_u^r)$ are consistent.*

The conventions taken for the local rules are so that integer z stands for the ‘successor number z ’. Hence the vertices designated by $\varepsilon, 1, 2 \dots$ are successors of the vertex ε , whereas those designated by $u.1, u.2 \dots$ are successors of its neighbour $u \in X^r$. For instance a vertex named $\{1, ab.2\}$ is understood to be both the first successor of vertex ε and the second successor of vertex ab . Such a vertex can be designated by $1, ab.2$ or $\{1, ab.2\}$.

Definition 32 (Localizable function). *A function F from $\mathcal{X}_{\Sigma,\Delta,\pi}$ to $\mathcal{X}_{\Sigma,\Delta,\pi}$ is said to be localizable if and only if there exists a radius r and a local rule f from $\mathcal{X}_{\Sigma,\Delta,\pi}^r$ to $\mathcal{G}_{\Sigma,\Delta,\pi}$ such that for all X , $F(X)$ is given by the equivalence class, with ε taken as the pointer vertex, of the graph*

$$\bigcup_{u \in X} u.f(X_u^r).$$

Equivalence. The following theorem shows that this constructive definition is in fact equivalent to the topological definition of causal functions.

Theorem 2 (Causal is localizable). *Let F be a function from $\mathcal{X}_{\Sigma,\Delta,\pi}$ to $\mathcal{G}_{\Sigma,\Delta,\pi}$. The function F is localizable if and only if there exists R_\bullet such that (F, R_\bullet) is a causal dynamics.*

Proof. [**Loc.** \Rightarrow **Caus.**] Let $F : \mathcal{X}_{\Sigma,\Delta,\pi} \rightarrow \mathcal{X}_{\Sigma,\Delta,\pi}$ be a localizable dynamics with local rule f from $\mathcal{X}_{\Sigma,\Delta,\pi}^r$ to $\mathcal{G}_{\Sigma,\Delta,\pi}$: $F(X)$ is the equivalence class, with ε taken as the pointer vertex, of the graph $H(X) = \bigcup u.f(X_u^r)$.

[Dynamics] Using the dynamicity of the local rule f , for all X^r we have $\varepsilon \in f(X^r)$. Therefore for all $u \in X$, we have $u \in u.f(X_u^r)$ and thus $u \in H(X)$. Let R be an isomorphism such that $G(F(X)) = RH(X)$. Let $u \in V(X)$, we define $R_X(u)$ to be $R(u')$, where u' is the vertex of $H(X)$ such that contains u in its name. Notice that $\sim(H(X), u) = \sim(R_X H(X), R_X(u)) = \sim(G(F(X)), R_X(u)) = F(X)_{R_X(u)}$.

[Translation-invariance] Take $u \in X$. We have $H(X_u) = \bigcup v.f(X_{u,v}^r)$. This is equal to $H(X_u) = \bar{u} \cdot \bigcup u.v.f(X_{u,v}^r)$, which in turn is equal to $\bar{u} \cdot H(X)$. Next, we have that $F(X_u) = \sim(H(X_u), \varepsilon) = \sim(\bar{u} \cdot H(X), \bar{u} \cdot u) = \sim(H(X), u) = F(X)_{R_X(u)}$. It follows that $F(X_u) = F(X)_{R_X(u)}$, and so $G(F(X_u)) = \overline{R_X(u)} \cdot G(F(X))$. We have therefore

$$G(F(X)) = R_X(u) \cdot G(F(X_u)) = R_X(u) \cdot R_{X_u} H(X_u) = R_X(u) \cdot R_{X_u} \bar{u} \cdot H(X).$$

But since the relation $G(F(X)) = RH(X)$ defines R_X , we have proven that for all $u \in X$, $R_X = (R_X(u) \cdot R_{X_u} \bar{u})$. It follows that, for all $u.v \in X$, $R_X(u.v) = R_X(u) \cdot R_{X_u}(v)$.

[Boundedness] for all X , for all $w' \in F(X)$, consider $w \in H(X)$ such that $w' = R(w)$ when $G(F(X)) = RH(X)$, and $u \in X$ such that $w \in u.f(X_u^r)$. Since $\varepsilon \in f(X_u^r)$, we have $u \in u.f(X_u^r)$. Since f is bounded, w lies at most at a some distance b of u in $H(X)$. Since $G(F(X)) = RH(X)$, w' lies at most at a some distance b of $u' = R(u) = R_X(u)$ in $F(X)$.

[Continuity] Let $m \in \mathbb{N}$. We must show that there exists n such that $F(X)^m = \tilde{H}(X)_\varepsilon^m$ is determined by X^n .

Consider a sequence $v_0 = \varepsilon, v_1, \dots, v_{m+1}$ of vertices of $H(X)$ such that for all $i \in \{0 \dots m\}$ there exists $e_i = (v_i : a_i, v_{i+1} : b_{i+1})$ in $E(H(X))$. For such an e_i to exist, and given Definitions 19 and 20, it must appear in some $u_i.f(X_{u_i}^r)$. Moreover if $\delta(e_i)$ is defined, it must be defined in some $u_i.f(X_{u_i}^r)$. Consider u_0, u_1, \dots, u_m a sequence of vertices of X such that this is the case. Also, since v_{i+1} is a subset of $V(X) \cdot S$, there exists $w_i \in X, z_i \in S$ such that $w_i \cdot z_i \in v_i$. Again consider $w_0 = \varepsilon, w_1, \dots, w_{m+1}$ a sequence of vertices of X such that this is the case.

Since e_i is in $u_i.f(X_{u_i}^r)$, it follows that v_i and v_{i+1} are in $u_i.f(X_{u_i}^r)$. This entails that v_i and v_{i+1} are subsets of $u_i \cdot V(X_{u_i}) \cdot S$, thus in particular $w_i, w_{i+1} \in u_i \cdot V(X_{u_i})$. Therefore we have both $w_{i+1} \in u_i \cdot X_{u_i}$ and $w_{i+1} \in u_{i+1} \cdot X_{u_{i+1}}$. As a consequence u_i and u_{i+1} lie at distance $2(r+1)$ in X , and it follows that $\bigcup_{i=0 \dots m} u_i \cdot X_{u_i}^r \subseteq X^{2(m+1)(r+1)-1}$. Hence $X^{2(m+1)(r+1)-1}$ determines $E(H(X)_\varepsilon^m)$

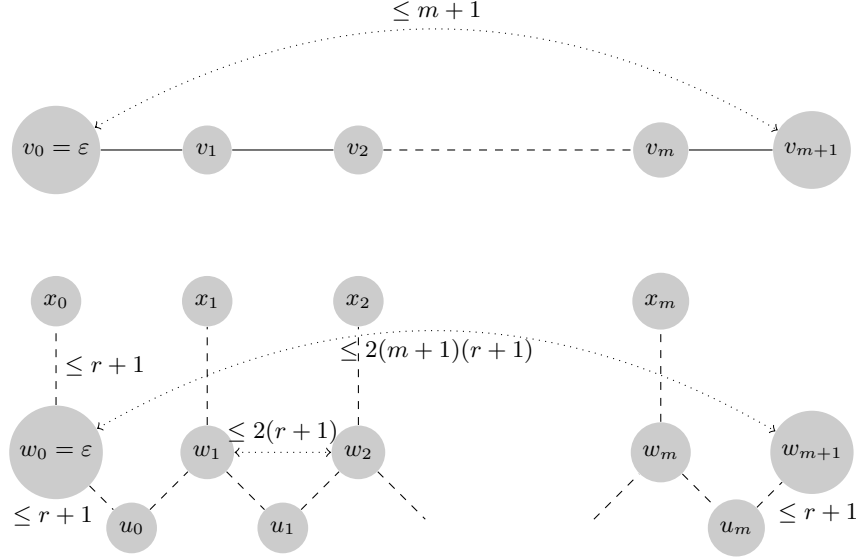


Fig. 1. Proof of continuity.

and their internal states.

For $\sigma(v_i)$ to be defined, there must exist $x_i \in X$ such that $\sigma(v_i)$ is defined in $x_i.f(X_{x_i}^r)$. Consider x_0, x_1, \dots, x_m a sequence of vertices of X such that this is the case. But since $v_i \in x_i.f(X_{x_i}^r)$, we must have that $w_i \in x_i.X_{x_i}^r$. Thus x_{j+1} lies at distance at most $r+1$ of $u_j.X_{u_j}^r$. Hence x_j lies at distance at most $r+1$ of $\bigcup_{i=0}^{m-1} u_i.X_{u_i}^r \subseteq X^{2m(r+1)-1}$. Hence $x_j \in X^{2m(r+1)+r}$, and thus $\bigcup_{i=0}^m x_i.X_{x_i}^r \subseteq X^{2m(r+1)+2r+1}$. Hence $X^{2(m+1)(r+1)-1}$ determines the internal states of $H(X)_\epsilon^m$.

Summarizing, X^n , with $n = 2(m+1)(r+1) - 1$ determines $F(X)^m = \tilde{H}(X)_\epsilon^m$. Consider some $v'' \in R_X^m$. This means that $v'' \in (RH(X))_\epsilon^m$ and $v'' = R(v')$ for some $v' \in H(X)$ that contains $v \in X$ in its name. Hence $v' \in H(X)_\epsilon^m$, where we used $R(\epsilon) = \epsilon$. Since this is determined by X^n , we have $v \in X^n$. Hence $\text{dom} R_X^m \subseteq X^n$. Moreover, consider X' such that $X'^r = X^r$. Therefore $v \in X'^r$, $H(X)_\epsilon^m$ and $H(X')_\epsilon^m$ are isomorphic, and this isomorphism sends v' to the w' of $H(X')_\epsilon^m$ whose name contains v . Therefore $F(X)^m$ and $F(X')^m$ are equal, and the same paths designate $R_X^m(v)$ and $R_{X'}^m(v)$, which are thus equal.

[Caus. \Rightarrow Loc.] Let (F, R_\bullet) be a causal dynamics. Let b_0 and b_1 be respectively the bounds given by Definition 27 and Lemma 3, and $b = \max(b_0 + 1, b_1)$. Let $m = 3b + 2$. Let r the radius such that for all $X, X', X^r = X'^r$ implies $F(X)^m = F(X')^m$ and $R_X^m = R_{X'}^m$, from Definition 28 and Theorem 3. We will construct f from X^r to $\mathcal{G}_{\Sigma, \Delta, \pi}$ so that for all X^r , the graph $f(X^r)$ is a well-chosen member of the equivalence class $F(X^r)^b$. Hence we must instantiate $F(X^r)^b$ via a suitable, local naming of its vertices. We use the isomorphism S_{X^r} of Lemma 4 for this purpose, i.e. $f(X^r) = S_{X^r}G(F(X^r)^b)$.

[Dynamics] For all X^r , $f(X^r)$ has vertices that are subsets of $V(X^r).S$, by definition. These sets are disjoint, by Lemma 4 (i) applied to pairs of vertices of $F(X^r)^b$. Moreover $\varepsilon \in f(X^r)$, since $\varepsilon \in F(X^r)^b$ and $S_{X^r}(\varepsilon) = \varepsilon$ by Lemma 4 (ii).

[Boundedness] For all X^r , the graph $f(X^r)$ is finite, by construction.

[Consistency] In order to show the consistency of f , we will show that for all X , $u \in X$, we have that $u.f(X_u^r)$ is the subgraph $H(X)_u^b$ of $H(X)$, where $H(X)$ is a well-chosen member of the equivalence class $F(X)$. Hence we must instantiate $F(X)$ via a suitable naming of its vertices. We use the isomorphism S_X of Lemma 4 for this purpose, i.e. $H(X) = S_X G(F(X))$.

Start from $u.f(X_u^r) = u.S_{X_u^r} G(F(X_u^r)^b)$, which is equal to $u.S_{X_u} G(F(X_u)^b)$, by Lemma 4 (iii) and using the fact that $F(Y)^b = F(Y^r)^b$. This, in turn, is equal to $u.(S_{X_u} G(F(X_u)))^b$, using the natural inclusion of $F(Y)^b$ into $F(Y)$. This, in turn, is equal to $u.(S_{X_u} \overline{R_X(u)}.G(F(X)))^b$, by shift-invariance, which is equal to $u.(\overline{u}.S_X G(F(X)))^b$, by Lemma 4 (iv). This, finally, is $(S_X G(F(X)))_u^b = H(X)_u^b$, since it is true that for any graph G and any isomorphism T , $TG_u^b = (TG)_{T(u)}^b$ and thus $G_u^b = T^{-1}(TG)_{T(u)}^b$.

Summarizing, $u.f(X_u^r) = H(X)_u^b$. Moreover if $u \in X^0$, then notice that $u \in f(X^r)$ and $u \in u.f(X_u^r)$, and hence they are non-trivially consistent.

Since f is consistent, and $f(X^r)$ is a representant of $F(X^r)^b$, it remains only to remark that $F(X) = \bigcup u.F(X_u^r)^b$, which is true because b was chosen to be strictly larger the one given by Definition 27, insuring that all the vertices and edges of $F(X^r)$ are covered, along with their labels.

Lemma 4 (Local renaming properties). *Let (F, R_\bullet) be a causal dynamics. Let b be the maximum of the bounds from Definition 27 and Lemma 3. Let $m = 3b + 2$. Let r the radius such that for all X, X' , $X^r = X'^r$ implies $F(X)^m = F(X')^m$ and $R_X^m = R_{X'}^m$, from Definition 28 and Proposition 3. Let z be an injection from $V(X_\pi^b) \setminus \varepsilon$, as in Definition 14, to \mathbb{N} . Let $z(\varepsilon)$ be the empty word. Let Y be a generalized Cayley graph. Consider S_Y such that for all $w' \in F(Y)$ we have*

$$S_Y(w') = \{u.z(v') \mid u'.v' = w' \wedge u \in Y \wedge u' = R_Y(u) \wedge v' \in F(Y)_{u'}^b\}.$$

We have:

- (i) $\forall w'_1, w'_2 \in F(Y)$, $S_Y(w'_1) \cap S_Y(w'_2) \neq \emptyset \Rightarrow S_Y(w'_1) = S_Y(w'_2)$.
- (ii) $\varepsilon \in S_Y(\varepsilon)$.
- (iii) $\forall w' \in F(X_u)^b$, $u.S_{X_u^r}(w') = u.S_{X_u}(w')$.
- (iv) $\forall v' \in F(X_u)$, $S_X(R_X(u).v') = u.S_{X_u}(v')$.

Proof. [(i)] Consider w'_1, w'_2 such that $S_Y(w'_1)$ and $S_Y(w'_2)$ have a common element $u.z(v')$. This entails that $w'_1 = u'.v' = w'_2$ is the same vertex in $F(Y)$, and thus that $S_Y(w'_1) = S_Y(w'_2)$.

[(ii)] Since $z(\varepsilon) = \varepsilon$, $\varepsilon.\varepsilon = \varepsilon$, $\varepsilon = R_Y(\varepsilon)$ and $\varepsilon \in F(Y)^b$.

[(iii)] Consider the $u = \varepsilon$ case. Let w' be a vertex of $F(X)^b$, and $u' \in F(X)$

a vertex such that $u'.v' = w'$, with $|v'| \leq b + 1$. We necessarily have that $u' \in F(X)^{2b+1}$. Moreover, since $F(X)^{3b+2} = F(X^r)^{3b+2}$, we have $F(X)_{u'}^b = F(X^r)_{u'}^b$. Also, using $R_X^m = R_{X^r}^m$, we have that

$$u' = R_X(u) \Leftrightarrow u' = R_X^m(u) \Leftrightarrow u' = R_{X^r}^m(u) \Leftrightarrow u' = R_{X^r}(u).$$

where the middle equivalence uses the natural inclusion of X^r into X . As a consequence the two sets:

$$\begin{aligned} S_X(w') &= \{u.z(v') \mid u'.v' = w' \wedge u \in X \wedge u' = R_X(u) \wedge v' \in F(X)_{u'}^b\} \\ S_{X^r}(w') &= \{u.z(v') \mid u'.v' = w' \wedge u \in X^r \wedge u' = R_{X^r}(u) \wedge v' \in F(X^r)_{u'}^b\} \end{aligned}$$

are equal, up to the natural inclusion of X^r into X . The same holds for S_{X_u} and $S_{X_u^r}$. Then, since the shift operation $(u.)$ is from $V(X^n)$ to $V(X)$, full equality holds between $u.S_{X_u}$ and $u.S_{X_u^r}$.

[(iv)] Consider some $u'.v'.w' \in F(X)$ with $u' = R_X(u)$, $v' = R_{X_u}(v)$ and $w' \in F(X_{u.v})^b$.

$$\begin{aligned} u.S_{X_u}(v'.w') &= u.\{x.z(y') \mid v'.w' = x'.y' \wedge x \in X_u \\ &\quad \wedge x' = R_{X_u}(x) \wedge y' \in F(Y)_{u'}^b\} \\ &= \{u.x.z(y') \mid u'.v'.w' = u'.x'.y' \wedge u.x \in X \\ &\quad \wedge u'.x' = R_X(u.x) \wedge y' \in F(Y)_{u'}^b\} \\ &= S_X(u'.v'.w') \\ &= S_X(R_X(u).v'.w') \end{aligned}$$

Our causal dynamics over generalized Cayley graphs is a candidate model of computation accounting for space, but without this space being fixed. As a candidate model of computation, we must check that it is computable. The following shows that we can decide whether a syntactic object is a valid instance of the model.

Proposition 4 (Decidability of consistency). *Given a dynamics f from $\mathcal{X}_{\Sigma, \Delta, \pi}^r$ to $\mathcal{G}_{\Sigma, \Delta, \pi}$, it is decidable whether f is a local rule.*

Proof. First of all notice that there is a finite number of disks X^b of radius b , with labels in finite sets Δ and Σ . The following informal procedure verifies that f is a local rule:

- For each X^r check that $\varepsilon \in f(X^r)$.
- For each X^{r+1} check that for all $u \in X^0$, $f(X^r)$ and $u.f(X_u^r)$ are non-trivially consistent.
- For each X^{3r+2} check that for all $u \in X^{2r+1}$, $f(X^r)$ and $u.f(X_u^r)$ are non-trivially consistent.

Finally, we prove that if the initial state is finite, its evolution can be computed.

Proposition 5 (Computability of causal functions). *Given a local rule f and a finite generalized Cayley graph X , then $F(X)$ is computable, with F the causal dynamics induced by f .*

Proof. Since f is a local rule, the images of disks of radius r included in X are all finite, and consistent with one another. Moreover the finite union of finite, consistent graphs, is computable.

5 Examples

Turtle. Our first example describes a simple oscillating pattern. It acts on graphs of maximal degree one by switching between a pair of vertices and an isolated vertex, and is trivially causal. Vertices and edges are unlabelled. The local rule which implements this dynamics is described in Figure 2. This local rule is of radius zero, as it only needs to look at a vertex and its edges, and hence the directly neighbouring vertices.

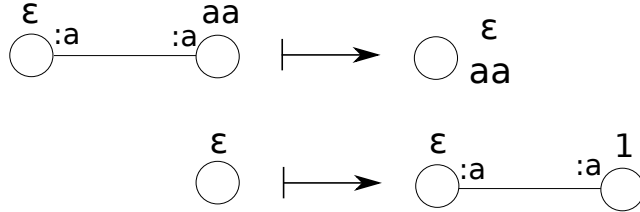


Fig. 2. The local rule describing the “turtle” dynamics.

Inflating grid. Our second example is taken from [1]. Each vertex gives birth to four distinct vertices, such that the structure of the initial graph is preserved, but inflated. The graph has maximal degree 4, as the ports take their names in $\pi = \{n, s, e, w\}$. Vertices and edges are unlabelled. The local rule is again of radius zero. The standard case of the local rule is described in Figure 3. It generates a subgraph of 8 vertices, with names serving as identification information, so that the generated graphs glue back together. For a complete definition we would also have to include the border cases, when vertex ε is surrounded by less than 4 neighbours (see Figure 4, for instance). Moreover, we would also have to include the cases when ports have not been set up in the natural manner (e.g. if there is an ee connection), or when the vertex has a self-edge, etc. None of these is a problem.

The global dynamics is obtained by glueing the different $u.f(X_u^r)$, identifying the vertices having the same set of names, as illustrated in Figure 5.

Lattice gas automaton. Our third example is an implementation of a one dimensional lattice gas automaton, taken from [9]. This dynamics describes the

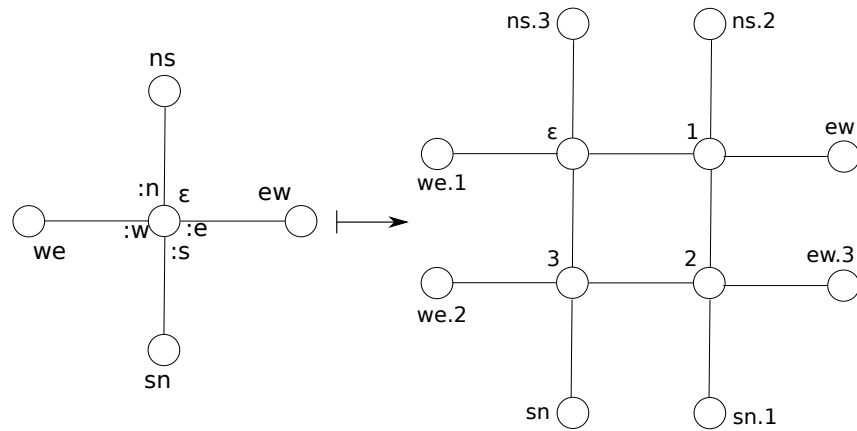


Fig. 3. Standard case of the local rule for the inflating grid (four neighbours)

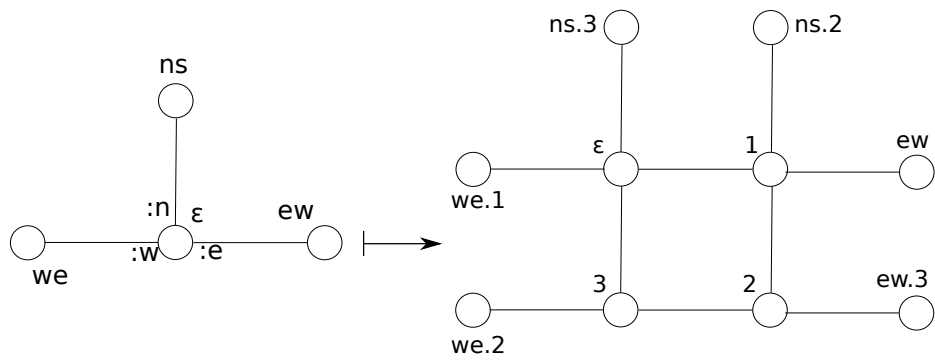


Fig. 4. Border case of the local rule for the inflating grid (three neighbours)

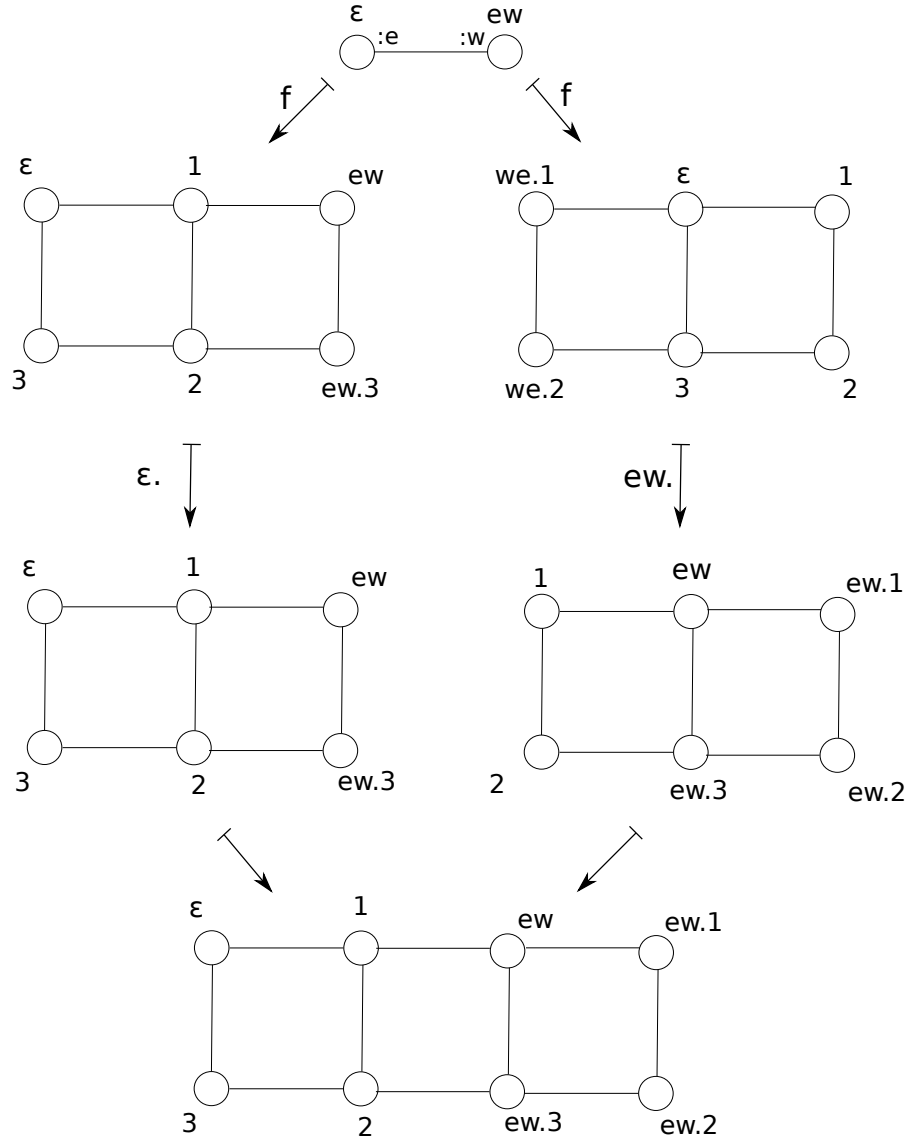


Fig. 5. The dynamics applied upon a pair of vertices. First the local rule is applied at each vertex. Then its images are shifted by the names of the vertices. Finally, they are glued back together.

propagation of particles along a line or a circle. Thus it acts on graphs of maximal degree two. Vertices are labelled to account for the presence or absence of a particle as well as its velocity direction (either left or right). Here the local rule is of radius two, but in our Figures only the significant vertices are shown. In addition to the intuitive propagation rules of Figure 6, two sets of rules are added, which modify the graph. These are the collapse rules of Figure 8, which delete a vertex, and the inflation rules of Figure 7, which create a vertex. These are triggered when two particles cross each other. Whether that deletes or adds a vertex to the graph, depends upon the way they cross (i.e. whether they were about to go past each other or to land on the same vertex). Again a great

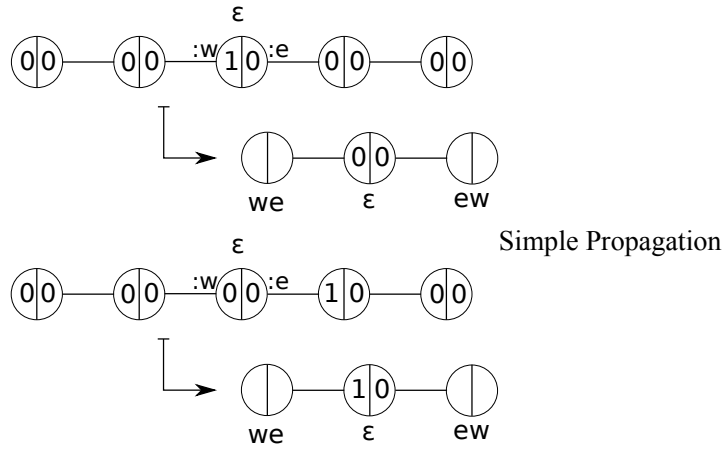


Fig. 6. Local rule of the Lattice Gas Automaton: propagation. In this drawing the internal state "(1|0)" means (1, 0), i.e. a signal moving east and none moving right, but if the ports of the vertex had been the other way round then it would have meant (0, 1).

deal of cases of the local rule have not been shown here. To be complete we would again have to include the cases when ports have not been set up in the natural manner (e.g. if there is an ee connection), or when the graphs a cycle, or when the number of particles is zero, three, etc. None of these is a problem. For instance, in [9], when there is more than two particles, the collapse and inflation rule do not apply: they simply propagate. The same would have to be done here.

6 Conclusion

Summary. First we have shown that a notion of graphs with ports modulo isomorphism (Definitions 1–5) provides a generalization of Cayley graphs, in the following sense: each vertex can be named relatively to the origin; each graph

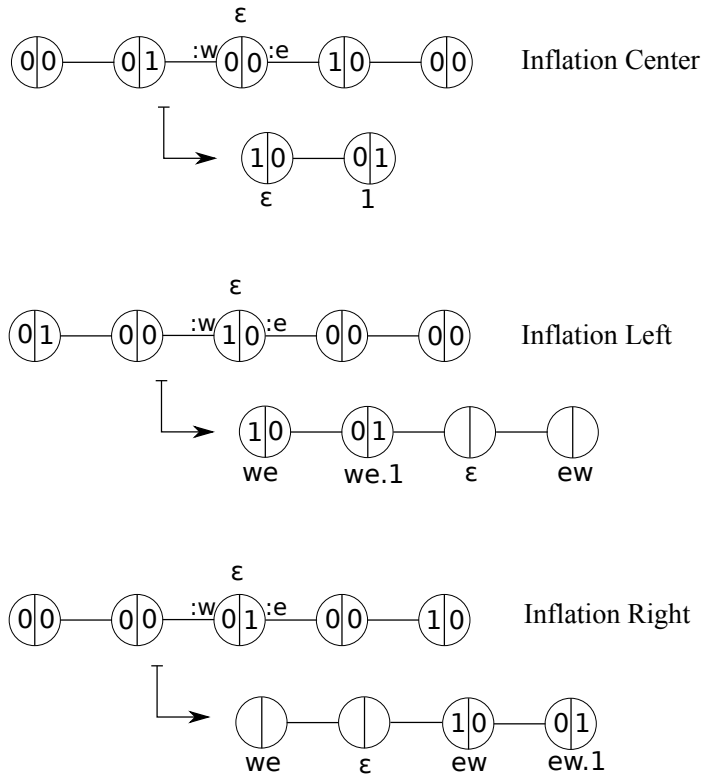


Fig. 7. Local rule of the Lattice Gas Automaton: inflation.

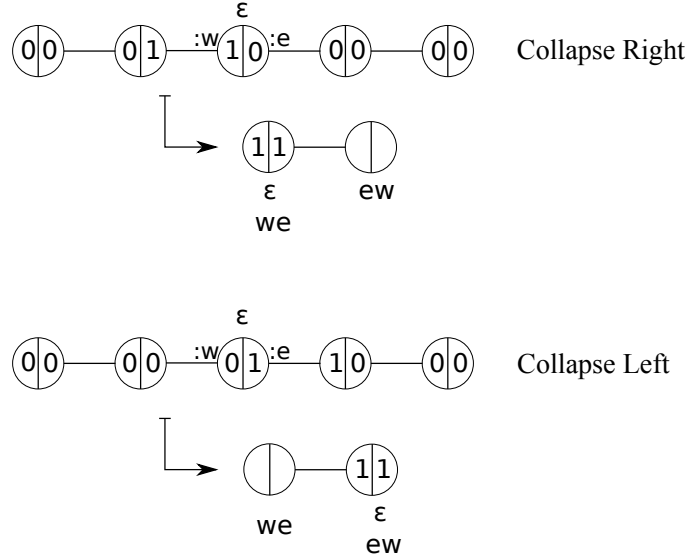


Fig. 8. Local rule of the Lattice Gas Automaton: collapse.

represents a language and its equivalence relation (Definitions 9–10, Theorem 1); and they are equipped with a well-defined notion of translation (Definition 15). Second, we have shown that the set of these graphs forms a compact metric space (Definition 21 and Lemma 2), entailing that continuous functions over this set are also uniformly continuous (Proposition 3). Third, this allowed us to characterize Cellular Automata over those generalized Cayley graphs as the set of shift-invariant, continuous, bounded dynamics (Definitions 24–28). This physically-motivated mathematical definition would have remained excessively abstract without our main result, showing that such causal dynamics are necessarily localizable, i.e. that they can be expressed as the synchronous, homogeneous application of a local rule (Definitions 29–32, Theorem 2). Finally, we showed that the property of being a local rule is decidable and hence that causal dynamics are computable (Propositions 4–5).

Further works. The mathematical relation between the causal dynamics of [1] and ours remains to be clarified – for instance, decidability remains to be proven for the causal graph dynamics of [1]. Still, they are important features of models of computation. The fact that they are relatively straightforward to prove in this paper is a good indicator that the formalism presented is appropriate. Our short terms plan, however, include: interpreting CA over generalized Cayley graphs as a dynamics over simplicial complexes on the one hand, and studying the reversible case on the other hand. Moreover it would be great to benefit from

the expertise of the Graph Rewriting community in order to design a convenient, succinct and powerful language for the description of the local rules.

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